

Energy level displacement of the excited $n\ell$ state of pionic hydrogen

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Abstract. The energy level displacements of the excited $n\ell$ states of pionic hydrogen and the contribution of the $ns \rightarrow 1s$ transitions and the $(\pi^- p)_{\text{Coul}} \rightarrow 1s$ transitions of the $\pi^- p$ pair, coupled by the attractive Coulomb field in the S -wave state with a continuous energy spectrum, to the shift of the energy level of the ground state of pionic hydrogen, caused by strong low-energy interactions, are calculated within a quantum field theoretic, relativistic covariant and model-independent approach.

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1 Introduction

Pionic hydrogen $A_{\pi p}$, a hydrogen-like mesoatom with the electron replaced by the π^- meson, is a nice laboratory for the experimental investigation of strong low-energy interactions and the mechanisms of spontaneous breaking of chiral symmetry [1, 2].

As has been found by Deser, Goldberger, Baumann and Thirring [3] (see also [4–8] and the textbook by Ericson and Weise [9]) due to strong low-energy interactions the energy level of the ground state of pionic hydrogen acquires the following shift and width:

$$\begin{aligned} -\epsilon_{1s} + i \frac{\Gamma_{1s}}{2} &= \frac{2\pi}{\mu} \left[\frac{1}{3} \left(2a_0^{1/2} + a_0^{3/2} \right) \right. \\ &\quad \left. + i \frac{2}{9} Q_{\pi^0 n} \left(a_0^{1/2} - a_0^{3/2} \right)^2 \right] |\Psi_{1s}(0)|^2 = \\ &= \frac{2\pi}{\mu} f_0^{\pi^- p}(0) |\Psi_{1s}(0)|^2. \end{aligned} \quad (1.1)$$

This is the so-called DGBT formula, $f_0^{\pi^- p}(0)$ is the S -wave amplitude of low-energy $\pi^- p$ scattering calculated at

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threshold, $a_0^{1/2}$ and $a_0^{3/2}$ are the S -wave scattering lengths of πN scattering with isospin $I = 1/2$ and $I = 3/2$, $Q_{\pi^0 n} = 28.040 \text{ MeV}$ is the relative momentum of the $\pi^0 n$ pair at relative momentum zero of the $\pi^- p$ pair, and $\Psi_{1s}(0) = 1/\sqrt{\pi a_B^3}$ is the wave function of pionic hydrogen in the ground state at the origin $r = 0$ ¹. The imaginary part of the amplitude $f_0^{\pi^- p}(0)$ in (1.1) is defined by the inelastic channel $\pi^- + p \rightarrow \pi^0 + n$.

The DGBT formula (1.1) can be transcribed into an equivalent form [3] (see also [9])

$$\begin{aligned} \frac{\epsilon_{1s}}{E_{1s}} &= + \frac{4}{a_B} A_S^{\pi^- p \rightarrow \pi^- p}, \\ \frac{\Gamma_{1s}}{E_{1s}} &= - \frac{8}{a_B} \left(A_S^{\pi^- p \rightarrow \pi^0 n} \right)^2 Q_{\pi^0 n}, \end{aligned} \quad (1.2)$$

where $E_{1s} = -\alpha/2a_B = -\alpha^2\mu/2 = -3234.940 \text{ eV}$ is the binding energy of the ground state of pionic hydrogen and

$$\begin{aligned} A_S^{\pi^- p \rightarrow \pi^- p} &= \frac{1}{3} \left(2a_0^{1/2} + a_0^{3/2} \right), \\ A_S^{\pi^- p \rightarrow \pi^0 n} &= \frac{\sqrt{2}}{3} \left(a_0^{3/2} - a_0^{1/2} \right) \end{aligned} \quad (1.3)$$

are the S -wave scattering lengths of $\pi^- p$ scattering.

¹ Here $a_B = 1/\alpha\mu = 222.664 \text{ fm}$ is the Bohr radius, where $\alpha = e^2 = 1/137.036$ is the fine-structure constant and $\mu = m_{\pi^-} m_p / (m_{\pi^-} + m_p) = 121.497 \text{ MeV}$ is the reduced mass of the $\pi^- p$ pair calculated for $m_{\pi^-} = 139.570 \text{ MeV}$ and $m_p = 938.272 \text{ MeV}$ [10].

The theoretical accuracy of the DGBT formula with respect to next-to-leading-order corrections caused by strong low-energy interactions has been recently analysed in [11]. As has been shown the second-order correction to the shift of the energy level of the ground state relative to the first order makes up $(0.111 \pm 0.006)\%$. In turn, the derivation of the energy level displacement of the ground state of pionic hydrogen, carried out within a quantum field theoretic, relativistic covariant and model-independent approach [11], leads to the non-perturbative correction of order of 1%. Hence, strong low-energy interactions cannot compete with contributions of QCD isospin-breaking and electromagnetic interactions [12]. The predicted value of these corrections is $\delta_\epsilon = (-7.3 \pm 2.9)\%$ [12]².

Experimentally [1,2], the energy level displacement of the ground state $(\epsilon_{1s}, \Gamma_{1s})$ can be obtained by measuring the $np \rightarrow 1s$ transitions in pionic hydrogen for $n = 2, 3, 4$, where n is the principle quantum number of the bound np state of pionic hydrogen with the angular momentum $\ell = 1$. As a result, the energy level displacements $(\epsilon_{np}, \Gamma_{np})$ of the excited np states turn out to be entangled into the definition of $(\epsilon_{1s}, \Gamma_{1s})$.

Within a potential model approach, the shift of the energy level of the excited ns state of pionic hydrogen has been calculated by Trueman [4] and Ericson, Loiseau and Wycech [8], who have also given a systematic analysis of electromagnetic corrections, and of the excited $n\ell$ state by Partensky and Ericson [5], Lambert [6] and Deloff [7].

The main aims of this paper are i) to derive the general formula for the energy level displacement of the $n\ell$ excited state of pionic hydrogen in terms of the partial-wave scattering lengths of πN scattering within a quantum field theoretic, relativistic covariant and model-independent approach developed in [11] (see also [14]), ii) to give an analytical expression and a numerical value of the energy level displacement of the np excited state of pionic hydrogen, iii) to calculate the second-order correction to the shift of the energy level of the ground state of pionic hydrogen caused by the $ns \rightarrow 1s$ transitions and the $(\pi^- p)_{\text{Coul}} \rightarrow 1s$ transitions of the $\pi^- p$ pair, coupled by the attractive Coulomb field in the S -wave state with a continuous energy spectrum [15], and iv) to compare our results with those obtained in [4–9].

As has been shown in [11] (see also [14]), the energy level displacement of the ground state of pionic hydrogen can be represented in terms of the momentum integrals:

$$\begin{aligned}
 -\epsilon_{1s} + i \frac{\Gamma_{1s}}{2} &= \frac{1}{4m_{\pi^-} m_p} \int \frac{d^3 k}{(2\pi)^3} \frac{d^3 q}{(2\pi)^3} \\
 &\times \sqrt{\frac{m_{\pi^-} - m_p}{E_{\pi^-}(k) E_p(k)}} \sqrt{\frac{m_{\pi^-} - m_p}{E_{\pi^-}(q) E_p(q)}} \\
 &\times \Phi_{1s}^\dagger(\vec{k}) M(\pi^-(\vec{q})p(-\vec{q}, \sigma_p) \rightarrow \\
 &\pi^-(\vec{k})p(-\vec{k}, \sigma_p)) \Phi_{1s}(\vec{q}), \quad (1.4)
 \end{aligned}$$

² It is likely that the correction obtained in [11] is also included in that calculated by Gasser *et al.* [12] within Chiral Perturbation Theory (ChPT) [13]. The analysis of this question is in progress.

where $M(\pi^-(\vec{q})p(-\vec{q}, \sigma_p) \rightarrow \pi^-(\vec{k})p(-\vec{k}, \sigma_p))$ is the amplitude of $\pi^- p$ scattering for arbitrary relative momenta of the $\pi^- p$ pair, $E_{\pi^-} = \sqrt{k^2 + m_{\pi^-}^2}$ and $E_p(k) = \sqrt{k^2 + m_p^2}$ are the energies of the π^- -meson and the proton, $\sigma_p = \pm 1/2$ is the polarization of the proton, $\Phi_{1s}(\vec{k})$ is the wave function of the ground state of pionic hydrogen in the momentum representation normalized by

$$\int \frac{d^3 k}{(2\pi)^3} |\Phi_{1s}(\vec{k})|^2 = 1. \quad (1.5)$$

Near threshold the r.h.s. of (1.4) can be rewritten as follows:

$$\begin{aligned}
 -\epsilon_{1s} + i \frac{\Gamma_{1s}}{2} &= \frac{2\pi}{\mu} \int \frac{d^3 k}{(2\pi)^3} \frac{d^3 q}{(2\pi)^3} \\
 &\times \sqrt{\frac{m_{\pi^-} - m_p}{E_{\pi^-}(k) E_p(k)}} \sqrt{\frac{m_{\pi^-} - m_p}{E_{\pi^-}(q) E_p(q)}} \\
 &\times \Phi_{1s}^\dagger(\vec{k}) f_0^{\pi^- p}(\sqrt{kq}) \Phi_{1s}(\vec{q}), \quad (1.6)
 \end{aligned}$$

where we have set [16]

$$\begin{aligned}
 M(\pi^-(\vec{q})p(-\vec{q}, \sigma_p) \rightarrow \pi^-(\vec{k})p(-\vec{k}, \sigma_p)) &= \\
 8\pi(m_{\pi^-} + m_p) f_0^{\pi^- p}(\sqrt{kq}). \quad (1.7)
 \end{aligned}$$

Due to the wave functions $\Phi_{1s}^\dagger(\vec{k})$ and $\Phi_{1s}(\vec{q})$, the integrand of the momentum integrals in (1.4) and (1.6) is concentrated around $k \sim q \sim 1/a_B = 0.887 \text{ MeV}$. This justifies the application of the low-energy limit $k, q \rightarrow 0$ to the calculation of the amplitude of $\pi^- p$ scattering [11]. As a result, the r.h.s. of (1.6) reduces to the form of the DGBT formula with an additional non-perturbative correction of order of 1%, caused by the smearing of the wave function of the ground state of pionic hydrogen around the origin [11].

The paper is organized as follows. In sect. 2 we construct the wave function of the excited $n\ell$ state of pionic hydrogen following the prescription developed in [11]. In sects. 3 and 4 following [11] we calculate the shift and the width of the energy level of the excited $n\ell$ state of pionic hydrogen within a quantum field theoretic, relativistic covariant and model-independent approach developed in [11]. In sect. 5 we calculate analytically and give a numerical estimate of the energy level displacement of the excited np state of pionic hydrogen. The shift of the energy level of the np state is found in analytical and numerical agreement with the result obtained by Ericson and Weise within a potential model approach [9]. In sect. 6 we calculate the contribution of the $ns \rightarrow 1s$ transitions and the $(\pi^- p)_{\text{Coul}} \rightarrow 1s$ transitions of the $\pi^- p$ pair, coupled by the attractive Coulomb field in the S -wave state with a continuous energy spectrum, to the shift of the energy level of the ground state of pionic hydrogen induced by strong low-energy interactions. We find that this contribution relative to the DGBT shift makes up 0.076%. In the conclusion we discuss the obtained results. In the appendix we calculate the momentum integral defining the energy level displacement of the np state of pionic hydrogen.

2 Wave function of $n\ell$ state of pionic hydrogen

The wave function of pionic hydrogen in the $1s$ state has been defined as [11]

$$|A_{\pi p}^{(1s)}(\vec{P}, \sigma_p)\rangle = \frac{1}{(2\pi)^3} \int \frac{d^3 k_{\pi^-}}{\sqrt{2E_{\pi^-}(\vec{k}_{\pi^-})}} \frac{d^3 k_p}{\sqrt{2E_p(\vec{k}_p)}} \delta^{(3)}(\vec{P} - \vec{k}_{\pi^-} - \vec{k}_p) \times \sqrt{2E_A^{(1s)}(\vec{k}_{\pi^-} + \vec{k}_p)} \Phi_{1s}(\vec{k}_{\pi^-}) |\pi^-(\vec{k}_{\pi^-}) p(\vec{k}_p, \sigma_p)\rangle, \quad (2.1)$$

where $E_A^{(1s)}(\vec{P}) = \sqrt{M_A^{(1s)2} + \vec{P}^2}$ and \vec{P} are the total energy and the momentum of pionic hydrogen, $M_A^{(1s)} = m_p + m_{\pi^-} + E_{1s}$ is the mass of pionic hydrogen in the $1s$ bound state, $\sigma_p = \pm 1/2$ is the proton polarization; $\Phi_{1s}(\vec{k}_{\pi^-})$ is the wave function of the $1s$ state of pionic hydrogen in the momentum representation. It is normalized to unity (1.5). The wave function $|\pi^-(\vec{k}_{\pi^-}) p(\vec{k}_p, \sigma_p)\rangle$ we define as [11]

$$|\pi^-(\vec{k}_{\pi^-}) p(\vec{k}_p, \sigma_p)\rangle = c_{\pi^-}^\dagger(\vec{k}_{\pi^-}) a_p^\dagger(\vec{k}_p, \sigma_p) |0\rangle, \quad (2.2)$$

where $c_{\pi^-}^\dagger(\vec{k}_{\pi^-})$ and $a_p^\dagger(\vec{k}_p, \sigma_p)$ are creation operators of the π^- -meson with momentum \vec{k}_{π^-} and the proton with momentum \vec{k}_p and polarization $\sigma_p = \pm 1/2$. They satisfy standard relativistic covariant commutation and anti-commutation relations:

$$\begin{aligned} [c_{\pi^-}(\vec{k}'_{\pi^-}), c_{\pi^-}^\dagger(\vec{k}_{\pi^-})] &= (2\pi)^3 2E_{\pi^-}(\vec{k}_{\pi^-}) \delta^{(3)}(\vec{k}'_{\pi^-} - \vec{k}_{\pi^-}), \\ [c_{\pi^-}(\vec{k}'_{\pi^-}), c_{\pi^-}(\vec{k}_{\pi^-})] &= [c_{\pi^-}^\dagger(\vec{k}'_{\pi^-}), c_{\pi^-}^\dagger(\vec{k}_{\pi^-})] = 0, \\ \{a_p(\vec{k}'_p, \sigma'_p), a_p^\dagger(\vec{k}_p, \sigma_p)\} &= (2\pi)^3 2E_p(\vec{k}_p) \delta^{(3)}(\vec{k}'_p - \vec{k}_p) \delta_{\sigma'_p \sigma_p}, \\ \{a_p(\vec{k}'_p, \sigma'_p), a_p(\vec{k}_p, \sigma_p)\} &= \{a_p^\dagger(\vec{k}'_p, \sigma'_p), a_p^\dagger(\vec{k}_p, \sigma_p)\} = 0. \end{aligned} \quad (2.3)$$

The wave function (2.1) is normalized by

$$\langle A_{\pi p}^{(1s)}(\vec{P}', \sigma'_p) | A_{\pi p}^{(1s)}(\vec{P}, \sigma_p) \rangle = (2\pi)^3 2E_A^{(1s)}(\vec{P}) \delta^{(3)}(\vec{P}' - \vec{P}) \delta_{\sigma'_p \sigma_p}. \quad (2.4)$$

This is a relativistic covariant normalization of the wave function. The wave function (2.1) has been applied to the derivation of the energy level displacement of the ground state of pionic hydrogen within a quantum field theoretic, relativistic covariant and model-independent approach [11].

In analogy with the $1s$ state, we define the wave function of the $n\ell$ excited state of pionic hydrogen:

$$|A_{\pi p}^{(n\ell m)}(\vec{P}, \sigma_p)\rangle = \frac{1}{(2\pi)^3} \int \frac{d^3 k_{\pi^-}}{\sqrt{2E_{\pi^-}(\vec{k}_{\pi^-})}} \frac{d^3 k_p}{\sqrt{2E_p(\vec{k}_p)}} \delta^{(3)}(\vec{P} - \vec{k}_{\pi^-} - \vec{k}_p) \times \sqrt{2E_A^{(n\ell)}(\vec{k}_{\pi^-} + \vec{k}_p)} \Phi_{n\ell m}(\vec{k}_{\pi^-}) |\pi^-(\vec{k}_{\pi^-}) p(\vec{k}_p, \sigma_p)\rangle, \quad (2.5)$$

where $E_A^{(n\ell)}(\vec{P}) = \sqrt{M_A^{(n\ell)2} + \vec{P}^2}$ and \vec{P} are the total energy and the momentum of pionic hydrogen, $M_A^{(n\ell)} = m_p + m_{\pi^-} + E_{n\ell}$ and $E_{n\ell}$ are the mass and the binding energy of pionic hydrogen in the $n\ell$ bound state. $\Phi_{n\ell m}(\vec{k})$ is the wave function of the excited $n\ell$ state in the momentum representation, where n is the *principle quantum number*, ℓ is the angular momentum $\ell = 0, 1, \dots, n-1$ and m is the magnetic quantum number $m = 0, \pm 1, \dots, \pm \ell$.

The wave function of pionic hydrogen in the coordinate representation $\Psi_{n\ell m}(\vec{r})$ is equal to [17]

$$\begin{aligned} \Psi_{n\ell m}(\vec{r}) &= R_{n\ell}(r) Y_{\ell m}(\vartheta, \varphi) = \\ &= -\frac{2}{n^2} \sqrt{\frac{(n-\ell-1)!}{[(n+\ell)!]^3 a_B^3}} \left(\frac{2}{n} \frac{r}{a_B}\right)^\ell \\ &\times e^{-r/na_B} L_{n+\ell}^{2\ell+1}\left(\frac{2}{n} \frac{r}{a_B}\right) Y_{\ell m}(\vartheta, \varphi), \end{aligned} \quad (2.6)$$

where $L_{n+\ell}^{2\ell+1}(\rho)$ are the generalized Laguerre polynomials defined by [17]

$$\begin{aligned} L_{n+\ell}^{2\ell+1}(\rho) &= (-1)^{2\ell+1} \frac{(n+\ell)!}{(n-\ell-1)!} \rho^{-(2\ell+1)} \\ &\times e^\rho \frac{d^{n-\ell-1}}{d\rho^{n-\ell-1}} (\rho^{n+\ell} e^{-\rho}), \end{aligned} \quad (2.7)$$

and $Y_{\ell m}(\vartheta, \varphi)$ are spherical harmonics normalized by

$$\int d\Omega Y_{\ell' m'}^*(\vartheta, \varphi) Y_{\ell m}(\vartheta, \varphi) = \delta_{\ell' \ell} \delta_{m' m}, \quad (2.8)$$

where $d\Omega = \sin \vartheta d\vartheta d\varphi$ is a solid angle.

In the momentum representation the wave function $\Phi_{n\ell m}(\vec{k})$ of pionic hydrogen is determined as

$$\begin{aligned} \Phi_{n\ell m}(\vec{k}) &= \int d^3 x \Psi_{n\ell m}(\vec{r}) e^{-i\vec{k} \cdot \vec{r}} = \\ &= \int_0^\infty dr r^2 R_{n\ell}(r) \int d\Omega \sum_{\ell'=0}^\infty \sum_{m'=-\ell'}^{\ell'} 4\pi (-i)^{\ell'} j_{\ell'}(kr) \\ &\times Y_{\ell' m'}^*(\vartheta, \varphi) Y_{\ell' m'}(\vartheta_{\vec{k}}, \varphi_{\vec{k}}) Y_{\ell m}(\vartheta, \varphi) = \\ &= (-i)^\ell Y_{\ell m}(\vartheta_{\vec{k}}, \varphi_{\vec{k}}) 4\pi \int_0^\infty j_\ell(kr) R_{n\ell}(r) r^2 dr = \\ &= (-i)^\ell \sqrt{4\pi} \Phi_{n\ell}(k) Y_{\ell m}(\vartheta_{\vec{k}}, \varphi_{\vec{k}}), \end{aligned} \quad (2.9)$$

where $j_\ell(kr)$ are spherical Bessel functions [18], $Y_{\ell m}(\vartheta_k, \varphi_k)$ and $\Phi_{n\ell}(k)$ are spherical harmonics and radial wave functions in momentum space. The radial wave functions $\Phi_{n\ell}(k)$ are defined as

$$\Phi_{n\ell}(k) = \sqrt{4\pi} \int_0^\infty j_\ell(kr) R_{n\ell}(r) r^2 dr. \quad (2.10)$$

Now we are able to proceed to calculating the energy level displacement of the excited $n\ell$ state of pionic hydrogen.

3 Shift of energy level of excited $n\ell$ state

The shift of the energy level $\epsilon_{n\ell}$ of the excited $n\ell$ state of pionic hydrogen we define as [11]

$$\begin{aligned} \epsilon_{n\ell} = & -\frac{1}{2M_A^{(n\ell)}} \frac{1}{2\ell+1} \\ & \times \sum_{m=-\ell}^{\ell} \langle A_{\pi p}^{(n\ell m)}(0, \sigma_p) | \mathcal{L}_{\text{str}}(0) | A_{\pi p}^{(n\ell m)}(0, \sigma_p) \rangle, \end{aligned} \quad (3.1)$$

where $\mathcal{L}_{\text{str}}(x)$ is an effective total Lagrangian of strong low-energy interactions. For the quantum field theoretic and model-independent calculation of the shift of the energy level of the $n\ell$ state of pionic hydrogen we will not specify $\mathcal{L}_{\text{str}}(x)$ in terms of interpolating fields of the coupled mesons and baryons. We would like to emphasize that $\mathcal{L}_{\text{str}}(x)$ is a total *effective* Lagrangian accounting for all strong low-energy interactions. In other words, this *effective* Lagrangian defines the \mathbb{T}_{str} -matrix of strong low-energy interactions

$$\mathbb{T}_{\text{str}} = \int d^4x \mathcal{L}_{\text{str}}(x) \quad (3.2)$$

obeying the unitary condition [19,16]

$$\mathbb{T}_{\text{str}} - \mathbb{T}_{\text{str}}^\dagger = i\mathbb{T}_{\text{str}}^\dagger \mathbb{T}_{\text{str}}. \quad (3.3)$$

This means that the matrix element of the *effective* Lagrangian $\mathcal{L}_{\text{str}}(0)$ between the states $|\pi^- p\rangle$ defines a physical amplitude of $\pi^- p$ scattering [11]

$$\langle \pi^- p | \mathcal{L}_{\text{str}}(0) | \pi^- p \rangle = \frac{1}{3} \left(2T^{1/2} + T^{3/2} \right), \quad (3.4)$$

where $T^{1/2}$ and $T^{3/2}$ are the amplitudes of πN scattering with isotopic spin $I = 1/2$ and $I = 3/2$, respectively.

According to [11], the shift of the energy level of the $n\ell$ state, expressed in terms of the matrix element of the effective Lagrangian $\mathcal{L}_{\text{str}}(0)$, reads

$$\begin{aligned} \epsilon_{n\ell} = & -\frac{1}{2\ell+1} \sum_{m=-\ell}^{\ell} \int \frac{d^3k}{(2\pi)^3} \frac{\Phi_{n\ell m}^\dagger(\vec{k})}{\sqrt{2E_{\pi^-}(\vec{k})2E_p(\vec{k})}} \\ & \times \int \frac{d^3q}{(2\pi)^3} \frac{\Phi_{n\ell m}(\vec{q})}{\sqrt{2E_{\pi^-}(\vec{q})2E_p(\vec{q})}} \\ & \times \langle \pi^-(\vec{k}) p(-\vec{k}, \sigma_p) | \mathcal{L}_{\text{str}}(0) | \pi^-(\vec{q}) p(-\vec{q}, \sigma_p) \rangle. \end{aligned} \quad (3.5)$$

It is convenient to redefine the r.h.s. of (3.5) as follows:

$$\begin{aligned} \epsilon_{n\ell} = & -\frac{1}{2\ell+1} \sum_{m=-\ell}^{\ell} \int \frac{d^3k}{(2\pi)^3} \frac{\Phi_{n\ell}^\dagger(k)}{\sqrt{2E_{\pi^-}(k)2E_p(k)}} \\ & \times \int \frac{d^3q}{(2\pi)^3} \frac{\Phi_{n\ell}(q)}{\sqrt{2E_{\pi^-}(q)2E_p(q)}} \iint \frac{d\Omega_{\vec{k}}}{\sqrt{4\pi}} \frac{d\Omega_{\vec{q}}}{\sqrt{4\pi}} Y_{\ell m}^*(\vartheta_{\vec{k}}, \varphi_{\vec{k}}) \\ & \times \langle \pi^-(\vec{k}) p(-\vec{k}, \sigma_p) | \mathcal{L}_{\text{str}}(0) | \pi^-(\vec{q}) p(-\vec{q}, \sigma_p) \rangle Y_{\ell m}(\vartheta_{\vec{q}}, \varphi_{\vec{q}}). \end{aligned} \quad (3.6)$$

Since there is no spin-flip in the low-energy transition $\pi^- + p \rightarrow \pi^- + p$, the amplitude of $\pi^- p$ scattering is determined by [20,21]

$$\begin{aligned} & \langle \pi^-(\vec{k}) p(-\vec{k}, \sigma_p) | \mathcal{L}_{\text{str}}(0) | \pi^-(\vec{q}) p(-\vec{q}, \sigma_p) \rangle = 8\pi \sqrt{s} \\ & \times \sum_{\ell'=0}^{\infty} \left[(\ell'+1) f_{\ell'+}(\sqrt{kq}) + \ell' f_{\ell'-}(\sqrt{kq}) \right] P_{\ell'}(\cos \vartheta) = \\ & 8\pi \sqrt{s} \sum_{\ell'=0}^{\infty} \left[(\ell'+1) f_{\ell'+}(\sqrt{kq}) + \ell' f_{\ell'-}(\sqrt{kq}) \right] \\ & \times \sum_{m'=-\ell'}^{\ell'} \frac{4\pi}{2\ell'+1} Y_{\ell' m'}^*(\vartheta_{\vec{q}}, \varphi_{\vec{q}}) Y_{\ell' m'}(\vartheta_{\vec{k}}, \varphi_{\vec{k}}), \end{aligned} \quad (3.7)$$

where \sqrt{s} is the total energy in the s -channel of $\pi^- p$ scattering, $P_{\ell'}(\cos \vartheta)$ are Legendre polynomials [18] and ϑ is the angle between the relative momenta \vec{k} and \vec{q} . The amplitudes $f_{\ell'+}(\sqrt{kq})$ and $f_{\ell'-}(\sqrt{kq})$ describe $\pi^- p$ scattering in the states with a total momentum $J = \ell' + 1/2$ and $J = \ell' - 1/2$. They are defined by the phase shifts [20,21]

$$\begin{aligned} f_{\ell'+}(\sqrt{kq}) &= \frac{e^{+i\delta_{\ell'+}(\sqrt{kq})}}{\sqrt{kq}} \sin \delta_{\ell'+}(\sqrt{kq}), \\ f_{\ell'-}(\sqrt{kq}) &= \frac{e^{+i\delta_{\ell'-}(\sqrt{kq})}}{\sqrt{kq}} \sin \delta_{\ell'-}(\sqrt{kq}). \end{aligned} \quad (3.8)$$

Near threshold $k, q \rightarrow 0$ the amplitudes $f_{\ell'+}(\sqrt{kq})$ and $f_{\ell'-}(\sqrt{kq})$ are defined by the ℓ' -wave scattering lengths of $\pi^- p$ scattering [21]

$$\begin{aligned} f_{\ell'+}(\sqrt{kq}) &= a_{\ell'+}^{\pi^- p \rightarrow \pi^- p}(kq)^{\ell'}, \\ f_{\ell'-}(\sqrt{kq}) &= a_{\ell'-}^{\pi^- p \rightarrow \pi^- p}(kq)^{\ell'}. \end{aligned} \quad (3.9)$$

Substituting (3.7) and (3.9) in (3.6) and integrating over the solid angles, we get the shift of the energy level of the excited $n\ell$ state:

$$\begin{aligned} \epsilon_{n\ell} = & -\frac{2\pi}{\mu} \frac{(\ell+1) a_{\ell+}^{\pi^- p \rightarrow \pi^- p} + \ell a_{\ell-}^{\pi^- p \rightarrow \pi^- p}}{2\ell+1} \\ & \times \left| \int \frac{d^3k}{(2\pi)^3} \sqrt{\frac{m_{\pi^-} - m_p}{E_{\pi^-}(k)E_p(k)}} k^\ell \Phi_{n\ell}(k) \right|^2. \end{aligned} \quad (3.10)$$

This is a generalization of the DGBT formula to any excited $n\ell$ state of pionic hydrogen.

The ℓ -wave scattering lengths $a_{\ell\pm}^{\pi^-p \rightarrow \pi^-p}$ are related to the ℓ -wave scattering lengths $a_{\ell\pm}^I$, for $I = 1/2$ and $I = 3/2$ as

$$a_{\ell\pm}^{\pi^-p \rightarrow \pi^-p} = \frac{1}{3} \left(2a_{\ell\pm}^{1/2} + a_{\ell\pm}^{3/2} \right). \quad (3.11)$$

For the ground state $n = 1$ and $\ell = 0$ the expression (3.10) with (3.11) reduces to the DGBT formula (1.1) [11].

4 Width of energy level of excited $n\ell$ state

According to [11], the width $\Gamma_{n\ell}$ of the energy level of the excited $n\ell$ state is defined by

$$\begin{aligned} \Gamma_{n\ell} = & \frac{1}{2\ell+1} \sum_{m=-\ell}^{\ell} \int \frac{d^3k}{(2\pi)^3} \frac{\Phi_{n\ell m}^\dagger(\vec{k})}{\sqrt{2E_{\pi^-}(\vec{k})2E_p(\vec{k})}} \\ & \times \int \frac{d^3q}{(2\pi)^3} \frac{\Phi_{n\ell m}(\vec{q})}{\sqrt{2E_{\pi^-}(\vec{q})2E_p(\vec{q})}} \\ & \times \sum_{\lambda_n=\pm 1/2} \int \frac{d^3Q}{(2\pi)^3 2E_{\pi^0}(\vec{Q})2E_n(\vec{Q})} \\ & \times 2\pi \delta(E_{\pi^0}(\vec{Q}) + E_n(\vec{Q}) - m_{\pi^-} - m_p + E_{n\ell}) \\ & \times \langle \pi^-(\vec{k})p(-\vec{k}, \sigma) | \mathcal{L}_{\text{str}}(0) | \pi^0(\vec{Q})n(-\vec{Q}, \lambda_n) \rangle \\ & \times \langle n(-\vec{Q}, \lambda_n) \pi^0(\vec{Q}) | \mathcal{L}_{\text{str}}(0) | \pi^-(\vec{q})p(-\vec{q}, \sigma) \rangle. \end{aligned} \quad (4.1)$$

For the subsequent calculation it is convenient to rewrite the r.h.s. of (4.1) as follows:

$$\begin{aligned} \Gamma_{n\ell} = & \frac{1}{16\pi} \frac{1}{2\ell+1} \sum_{m=-\ell}^{\ell} \int \frac{d^3k}{(2\pi)^3} \frac{\Phi_{n\ell}^\dagger(k)}{\sqrt{E_{\pi^-}(k)E_p(k)}} \\ & \times \int \frac{d^3q}{(2\pi)^3} \frac{\Phi_{n\ell}(q)}{\sqrt{E_{\pi^-}(q)E_p(q)}} \sum_{\lambda_n=\pm 1/2} \int_0^\infty \frac{dQ Q^2}{E_{\pi^0}(Q)E_n(Q)} \\ & \times \delta(E_{\pi^0}(Q) + E_n(Q) - m_{\pi^-} - m_p - E_{n\ell}) \\ & \times \int \frac{d\Omega_{\vec{Q}}}{4\pi} \int \frac{d\Omega_{\vec{k}}}{\sqrt{4\pi}} \int \frac{d\Omega_{\vec{q}}}{\sqrt{4\pi}} Y_{\ell m}^*(\vartheta_{\vec{k}}, \varphi_{\vec{k}}) \\ & \times \langle \pi^-(\vec{k})p(-\vec{k}, \sigma) | \mathcal{L}_{\text{str}}(0) | \pi^0(\vec{Q})n(-\vec{Q}, \lambda_n) \rangle \\ & \times \langle n(-\vec{Q}, \lambda_n) \pi^0(\vec{Q}) | \mathcal{L}_{\text{str}}(0) | \pi^-(\vec{q})p(-\vec{q}, \sigma) \rangle Y_{\ell m}(\vartheta_{\vec{q}}, \varphi_{\vec{q}}). \end{aligned} \quad (4.2)$$

The matrix element $\langle \pi^-(\vec{k})p(-\vec{k}, \sigma) | \mathcal{L}_{\text{str}}(0) | \pi^0(\vec{Q})n(-\vec{Q}, \lambda_n) \rangle \times n(-\vec{Q}, \lambda_n)$ we define as

$$\begin{aligned} & \langle \pi^-(\vec{k})p(-\vec{k}, \sigma) | \mathcal{L}_{\text{str}}(0) | \pi^0(\vec{Q})n(-\vec{Q}, \lambda_n) \rangle = 8\pi \sqrt{s} \\ & \times \sum_{\ell'=0}^{\infty} \left[(\ell'+1) f_{\ell'+}(\sqrt{kQ}) + \ell' f_{\ell'-}(\sqrt{kQ}) \right] P_{\ell'}(\cos \vartheta) = \\ & 8\pi \sqrt{s} \sum_{\ell'=0}^{\infty} \left[(\ell'+1) f_{\ell'+}(\sqrt{kQ}) + \ell' f_{\ell'-}(\sqrt{kQ}) \right] \\ & \times \sum_{m'=-\ell'}^{\ell'} \frac{4\pi}{2\ell'+1} Y_{\ell' m'}^*(\vartheta_{\vec{Q}}, \varphi_{\vec{Q}}) Y_{\ell' m'}(\vartheta_{\vec{k}}, \varphi_{\vec{k}}), \end{aligned} \quad (4.3)$$

Near threshold the amplitudes $f_{\ell'+}(\sqrt{kQ})$ and $f_{\ell'-}(\sqrt{kQ})$ are defined by the ℓ' -wave scattering lengths of the $\pi^-p \rightarrow \pi^0n$ scattering [21]:

$$\begin{aligned} f_{\ell'+}(\sqrt{kQ}) &= a_{\ell'+}^{\pi^-p \rightarrow \pi^0n}(kQ)^{\ell'}, \\ f_{\ell'-}(\sqrt{kQ}) &= a_{\ell'-}^{\pi^-p \rightarrow \pi^0n}(kQ)^{\ell'}. \end{aligned} \quad (4.4)$$

The ℓ' -wave scattering lengths $a_{\ell'\pm}^{\pi^-p \rightarrow \pi^0n}$ are related to the ℓ' -wave scattering lengths $a_{\ell'\pm}^I$ for $I = 1/2$ and $I = 3/2$ as

$$a_{\ell'\pm}^{\pi^-p \rightarrow \pi^0n} = \frac{\sqrt{2}}{3} \left(a_{\ell'\pm}^{3/2} - a_{\ell'\pm}^{1/2} \right). \quad (4.5)$$

Substituting (4.3) and (4.4) in (4.2) and integrating over the solid angles and the phase volume we arrive at the expression

$$\begin{aligned} \Gamma_{n\ell} = & \frac{4\pi}{\mu} \left[\frac{(\ell+1) a_{\ell+}^{\pi^-p \rightarrow \pi^0n} + \ell a_{\ell-}^{\pi^-p \rightarrow \pi^0n}}{2\ell+1} \right]^2 Q_{n\ell}^{2\ell+1} \\ & \times \left| \int \frac{d^3k}{(2\pi)^3} \frac{m_{\pi^-} - m_p}{\sqrt{E_{\pi^-}(k)E_p(k)}} k^\ell \Phi_{n\ell}(k) \right|^2, \end{aligned} \quad (4.6)$$

where $Q_{n\ell}$ is a relative momentum of the π^0n pair

$$Q_{n\ell} = \sqrt{\frac{2m_{\pi^0}m_n}{m_{\pi^0} + m_n} (m_{\pi^-} + m_p - m_{\pi^0} - m_n + E_{n\ell})}, \quad (4.7)$$

where $m_{\pi^0} = 134.977$ MeV and $m_n = 939.565$ MeV [10].

This is a generalization of the DGBT formula to any excited $n\ell$ state of pionic hydrogen. For the ground state $n = 1$ and $\ell = 0$ we arrive at the DGBT formula (1.1).

5 Energy level displacement of excited np state

The analysis of experimental data obtained by the PSI Collaboration demands the knowledge of the energy level displacements of the excited np states. For $\ell = 1$ from formulas (3.10) and (4.6) one gets

$$\begin{aligned} \epsilon_{np} = & -\frac{2\pi}{\mu} \frac{1}{3} \left(2a_{P+}^{\pi^-p \rightarrow \pi^-p} + a_{P-}^{\pi^-p \rightarrow \pi^-p} \right) \\ & \times \left| \int \frac{d^3k}{(2\pi)^3} \frac{\sqrt{m_{\pi^-} - m_p}}{\sqrt{E_{\pi^-}(k)E_p(k)}} k \Phi_{np}(k) \right|^2 = \\ & -\frac{2\pi}{9} \frac{1}{\mu} \left[2 \left(2a_{P+}^{1/2} + a_{P-}^{1/2} \right) + \left(2a_{P+}^{3/2} + a_{P-}^{3/2} \right) \right] \\ & \times \left| \int \frac{d^3k}{(2\pi)^3} \frac{\sqrt{m_{\pi^-} - m_p}}{\sqrt{E_{\pi^-}(k)E_p(k)}} k \Phi_{np}(k) \right|^2, \end{aligned}$$

$$\begin{aligned}
\Gamma_{np} &= \frac{4\pi}{9} \frac{Q_{np}^3}{\mu} \left(2a_{P+}^{\pi^-p \rightarrow \pi^0 n} + a_{P-}^{\pi^-p \rightarrow \pi^0 n} \right)^2 \\
&\times \left| \int \frac{d^3k}{(2\pi)^3} \sqrt{\frac{m_{\pi^-} m_p}{E_{\pi^-}(k) E_p(k)}} k \Phi_{np}(k) \right|^2 = \\
&\frac{8\pi}{81} \frac{Q_{np}^3}{\mu} \left[\left(2a_{P+}^{3/2} + a_{P-}^{3/2} \right) - \left(2a_{P+}^{1/2} + a_{P-}^{1/2} \right) \right]^2 \\
&\times \left| \int \frac{d^3k}{(2\pi)^3} \sqrt{\frac{m_{\pi^-} m_p}{E_{\pi^-}(k) E_p(k)}} k \Phi_{np}(k) \right|^2, \quad (5.1)
\end{aligned}$$

where a_{P+}^I and a_{P-}^I are the P -wave scattering lengths of πN scattering with isospin I and total momentum $J = 3/2$ and $J = 1/2$, respectively [20,21].

The integral over k is calculated in the appendix. The result reads

$$\int \frac{d^3k}{(2\pi)^3} \sqrt{\frac{m_{\pi^-} m_p}{E_{\pi^-}(k) E_p(k)}} k \Phi_{np}(k) = \sqrt{\frac{n^2 - 1}{\pi n^5 a_B^5}}. \quad (5.2)$$

Substituting (5.2) in (5.1), we obtain the shift and the width of the energy level of the excited np state. They read

$$\begin{aligned}
\epsilon_{np} &= -\frac{2}{9} \frac{\alpha^5}{n^3} \left(1 - \frac{1}{n^2} \right) \left(\frac{m_{\pi^-} m_p}{m_{\pi^-} + m_p} \right)^4 \\
&\times \left[2 \left(2a_{P+}^{1/2} + a_{P-}^{1/2} \right) + \left(2a_{P+}^{3/2} + a_{P-}^{3/2} \right) \right], \\
\Gamma_{np} &\frac{8}{81} \frac{\alpha^5}{n^3} \left(1 - \frac{1}{n^2} \right) \left(\frac{m_{\pi^-} m_p}{m_{\pi^-} + m_p} \right)^4 \\
&\times \left[\left(2a_{P+}^{3/2} + a_{P-}^{3/2} \right) - \left(2a_{P+}^{1/2} + a_{P-}^{1/2} \right) \right]^2 \\
&\times \left[\frac{2m_{\pi^0} m_n}{m_{\pi^0} + m_n} (m_{\pi^-} + m_p - m_{\pi^0} - m_n + E_{np}) \right]^{3/2}. \quad (5.3)
\end{aligned}$$

The shift and the width of the energy level of the np excited state (5.3) can be rewritten in the equivalent form

$$\begin{aligned}
\frac{\epsilon_{np}}{E_{np}} &= +\frac{4}{n} \left(1 - \frac{1}{n^2} \right) \frac{A_P^{\pi^-p \rightarrow \pi^-p}}{a_B^3}, \\
\frac{\Gamma_{np}}{E_{np}} &= -\frac{8}{n} \left(1 - \frac{1}{n^2} \right) \frac{\left(A_P^{\pi^-p \rightarrow \pi^0 n} \right)^2}{a_B^3} Q_{np}^3, \quad (5.4)
\end{aligned}$$

where $E_{np} = -\alpha/2a_B n^2$ is the binding energy of the excited np state and

$$\begin{aligned}
A_P^{\pi^-p \rightarrow \pi^-p} &= \frac{1}{9} \left[2 \left(2a_{P+}^{1/2} + a_{P-}^{1/2} \right) + \left(2a_{P+}^{3/2} + a_{P-}^{3/2} \right) \right], \\
A_P^{\pi^-p \rightarrow \pi^0 n} &= \frac{\sqrt{2}}{9} \left[\left(2a_{P+}^{3/2} + a_{P-}^{3/2} \right) - \left(2a_{P+}^{1/2} + a_{P-}^{1/2} \right) \right] \quad (5.5)
\end{aligned}$$

are the P -wave scattering lengths of π^-p scattering [9]. The ratio ϵ_{np}/E_{np} in (5.4) is in analytical agreement with the result obtained by Ericson and Weise (see [9], eq. (6.29)).

In order to estimate the values of the shift and width of the excited np state we use the experimental data on the P -wave scattering lengths compiled in table 5.3 of ref. [21] (Höhler 78, input: Karlsruhe-Helsinki analysis 78):

$$\begin{aligned}
a_{P-}^{1/2} &= (-0.082 \pm 0.002) m_{\pi^-}^{-3}, \\
a_{P+}^{1/2} &= (-0.032 \pm 0.001) m_{\pi^-}^{-3}, \\
a_{P-}^{3/2} &= (-0.044 \pm 0.001) m_{\pi^-}^{-3}, \\
a_{P+}^{3/2} &= (+0.215 \pm 0.003) m_{\pi^-}^{-3}, \\
2 \left(2a_{P+}^{1/2} + a_{P-}^{1/2} \right) + \left(2a_{P+}^{3/2} + a_{P-}^{3/2} \right) &= (0.094 \pm 0.008) m_{\pi^-}^{-3}, \\
\left(2a_{P+}^{3/2} + a_{P-}^{3/2} \right) - \left(2a_{P+}^{1/2} + a_{P-}^{1/2} \right) &= (0.532 \pm 0.007) m_{\pi^-}^{-3}. \quad (5.6)
\end{aligned}$$

This yields

$$\begin{aligned}
\epsilon_{np} &= -\frac{1}{n^3} \left(1 - \frac{1}{n^2} \right) \times (3.47 \pm 0.30) \times 10^{-5} \text{ eV}, \\
\Gamma_{np} &= \frac{1}{n^3} \left(1 - \frac{1}{n^2} \right) \times (3.71 \pm 0.10) \times 10^{-7} \text{ eV}. \quad (5.7)
\end{aligned}$$

Thus, the values of the energy level displacements of the excited np states of pionic hydrogen are much smaller than 10^{-5} eV³. The experimental value of the energy level displacement of the ground state of pionic hydrogen is equal to [2]

$$\begin{aligned}
\epsilon_{1s}^{\text{exp}} &= -7.108 \pm 0.036 \text{ eV}, \\
\Gamma_{1s}^{\text{exp}} &= 0.868 \pm 0.054 \text{ eV} \quad (5.8)
\end{aligned}$$

with an accuracy about 0.5% and 6.2% for the shift and the width, respectively.

The level of accuracy in a new set of experiments is about 0.2%, *i.e.* ($\Delta_{\text{shift}}^{\text{exp}} = \pm 0.014$ eV), for the energy level shift and 1%, *i.e.* ($\Delta_{\text{width}}^{\text{exp}} = \pm 0.009$ eV), for the energy level width [1]. Hence, according to (5.7), the contributions of the energy level displacements of the excited np states to the transitions $np \rightarrow 1s$ with $n = 2, 3, 4$ can be neglected, since $|\Delta_{\text{shift}}^{\text{exp}}| \gg |\epsilon_{np}|$ and $|\Delta_{\text{width}}^{\text{exp}}| \gg \Gamma_{np}$.

6 Energy shift of ground state caused by $(\pi^-p)_{ns} \rightarrow (\pi^-p)_{1s}$ and $(\pi^-p)_{\text{Coul}} \rightarrow (\pi^-p)_{1s}$ transitions

In this section we calculate the shift of the energy level of the ground state of pionic hydrogen $\delta\epsilon_{1s}$ caused by the $(\pi^-p)_{ns} \rightarrow (\pi^-p)_{1s}$ and $(\pi^-p)_{\text{Coul}} \rightarrow (\pi^-p)_{1s}$ transitions induced by strong low-energy interactions, where $(\pi^-p)_{ns}$ is a bound ns state of the π^-p pair and $(\pi^-p)_{\text{Coul}}$ is the π^-p pair, coupled by the attractive Coulomb field in the

³ For pionium, the bound $\pi^- \pi^+$ state, the shift of the energy level of the np state has been recently calculated by Julia Schweizer [22] within Chiral Perturbation Theory [13].

$$\begin{aligned}
\delta\epsilon_{1s}^{(ds)} = & \sum_{\alpha_p=\pm 1/2} \mathcal{P} \int \frac{d^3k}{(2\pi)^3} \frac{\Phi_{1s}^*(k)}{\sqrt{2E_{\pi^-}(k)2E_p(k)}} \int \frac{d^3Q}{(2\pi)^3} \frac{\Phi_{1s}(Q)}{\sqrt{2E_{\pi^-}(Q)2E_p(Q)}} \\
& \times \frac{\langle \pi^-(\vec{k})p(-\vec{k}, \sigma_p) | \mathcal{L}_{\text{str}}(0) | p(-\vec{Q}, \sigma_p) \pi^-(\vec{Q}) \rangle \langle \pi^-(\vec{P})p(-\vec{P}, \sigma_p) | \mathcal{L}_{\text{str}}(0) | p(-\vec{q}, \sigma_p) \pi^-(\vec{q}) \rangle}{E_{\pi^-}(k) + E_p(k) - E_{\pi^-}(Q) - E_p(Q)} \\
& \times \int \frac{d^3P}{(2\pi)^3} \frac{\Phi_{1s}^*(P)}{\sqrt{2E_{\pi^-}(P)2E_p(P)}} \int \frac{d^3q}{(2\pi)^3} \frac{\Phi_{1s}(q)}{\sqrt{2E_{\pi^-}(q)2E_p(q)}} \\
& + \sum_{\alpha_p=\pm 1/2} \sum_{n=2}^{\infty} \mathcal{P} \int \frac{d^3k}{(2\pi)^3} \frac{\Phi_{1s}^*(k)}{\sqrt{2E_{\pi^-}(k)2E_p(k)}} \int \frac{d^3Q}{(2\pi)^3} \frac{\Phi_{ns}(Q)}{\sqrt{2E_{\pi^-}(Q)2E_p(Q)}} \\
& \times \frac{\langle \pi^-(\vec{k})p(-\vec{k}, \sigma_p) | \mathcal{L}_{\text{str}}(0) | p(-\vec{Q}, \sigma_p) \pi^-(\vec{Q}) \rangle \langle \pi^-(\vec{P})p(-\vec{P}, \sigma_p) | \mathcal{L}_{\text{str}}(0) | p(-\vec{q}, \sigma_p) \pi^-(\vec{q}) \rangle}{E_{\pi^-}(k) + E_p(k) + E_{1s} - E_{\pi^-}(Q) - E_p(Q) - E_{ns}} \\
& \times \int \frac{d^3P}{(2\pi)^3} \frac{\Phi_{ns}^*(P)}{\sqrt{2E_{\pi^-}(P)2E_p(P)}} \int \frac{d^3q}{(2\pi)^3} \frac{\Phi_{1s}(q)}{\sqrt{2E_{\pi^-}(q)2E_p(q)}}, \tag{6.3}
\end{aligned}$$

S -state with a continuous energy spectrum [15]. According to [11] the correction $\delta\epsilon_{1s}$ reads

$$\begin{aligned}
\delta\epsilon_{1s} = & -\frac{1}{2M_A^{(1s)}} \frac{i}{2} \int d^4x \langle A_{\pi p}^{(1s)}(\vec{P}, \sigma_p) \\
& \times |T(\mathcal{L}_{\text{str}}(x)\mathcal{L}_{\text{str}}(0))| A_{\pi p}^{(1s)}(\vec{P}, \sigma_p) \rangle \Big|_{\vec{P}=0}. \tag{6.1}
\end{aligned}$$

First, we consider the contribution of the discrete spectrum. For this aim, we use a unit operator which we define as

$$\begin{aligned}
\hat{1} = & \sum_{\alpha_p=\pm 1/2} \sum_{n=1}^{\infty} \sum_{\ell=0}^{n-1} \sum_{m=-\ell}^{\ell} \frac{1}{(2\pi)^3} \\
& \times \int \frac{d^3Q}{2E_A^{(n\ell)}(\vec{Q})} |A_{\pi p}^{(n\ell m)}(\vec{Q}, \alpha_p)\rangle \langle A_{\pi p}^{(n\ell m)}(\vec{Q}, \alpha_p)|. \tag{6.2}
\end{aligned}$$

Following [11] and using a unit operator (6.2) for the description of the intermediate states in (6.1), and integrating over angular degrees of freedom, we get

see eq. (6.3) above

where the abbreviation (ds) means the *discrete spectrum* and \mathcal{P} stands for the calculation of the principle value of the integral.

Due to the wave functions of pionic hydrogen the integrands in the r.h.s. of (6.3) can be taken at the low-energy limit [11]. This yields

$$\begin{aligned}
\delta\epsilon_{1s}^{(ds)} = & \frac{8\pi^2}{9} \frac{1}{\mu} \left(2a_0^{1/2} + a_0^{3/2}\right)^2 |\Psi_{1s}(0)|^2 \\
& \times \mathcal{P} \int \frac{d^3k}{(2\pi)^3} \frac{d^3Q}{(2\pi)^3} \frac{\Phi_{1s}^*(k)\Phi_{1s}(Q)}{k^2 - Q^2} + \frac{4\pi^2}{9} \\
& \times \frac{1}{\mu^2} \left(2a_0^{1/2} + a_0^{3/2}\right)^2 \sum_{n=2}^{\infty} \frac{|\Psi_{ns}^*(0)\Psi_{1s}(0)|^2}{E_{1s} - E_{ns}}. \tag{6.4}
\end{aligned}$$

Since in the momentum representation the wave function of the ground state of pionic hydrogen can be taken real,

$\Phi_{1s}^*(k) = \Phi_{1s}(k)$, the integrand of the first term in the r.h.s. of (6.4) is antisymmetric under the change of variables $k \longleftrightarrow Q$. Hence, the integral over \vec{k} and \vec{Q} should vanish.

Thus, the shift of the energy level of the ground state, caused by the $ns \rightarrow 1s$ transitions, is defined by

$$\delta\epsilon_{1s}^{(ds)} = \frac{4\pi^2}{9} \frac{1}{\mu^2} \left(2a_0^{1/2} + a_0^{3/2}\right)^2 \sum_{n=2}^{\infty} \frac{|\Psi_{ns}^*(0)\Psi_{1s}(0)|^2}{E_{1s} - E_{ns}}. \tag{6.5}$$

Setting $\Psi_{1s}(0) = 1/\sqrt{\pi a_B^3}$, $\Psi_{ns}(0) = 1/\sqrt{\pi n^3 a_B^3}$, $E_{1s} = -\alpha/2a_B$ and $E_{ns} = -\alpha/2a_B n^2$, we get

$$\begin{aligned}
\delta\epsilon_{1s}^{(ds)} = & -\frac{8}{9} \alpha^4 \mu^3 \left(2a_0^{1/2} + a_0^{3/2}\right)^2 \sum_{n=2}^{\infty} \frac{1}{n(n^2 - 1)} = \\
& -\frac{2}{9} \alpha^4 \mu^3 \left(2a_0^{1/2} + a_0^{3/2}\right)^2. \tag{6.6}
\end{aligned}$$

The contribution of the continuous spectrum of the π^-p pair, coupled by the attractive Coulomb field in the S -wave state, can be determined by

$$\delta\epsilon_{1s}^{(cs)} = \frac{4\pi^2}{9} \frac{1}{\mu^2} \left(2a_0^{1/2} + a_0^{3/2}\right)^2 |\Psi_{1s}(0)|^2 \int_0^{\infty} dE \frac{|\Psi_E(0)|^2}{E_{1s} - E}, \tag{6.7}$$

where the abbreviation (cs) means the *continuous spectrum*. The wave function $\Psi_E(\vec{r})$ of the π^-p pair, coupled by the attractive Coulomb fields in the S -wave state with a continuous energy spectrum, is equal to [17, 23]

$$\Psi_E(\vec{r}) = \sqrt{\frac{1}{4\pi} \frac{\alpha\mu^2}{1 - e^{-2\pi\alpha\mu/k}}} F\left(1 + i\frac{\alpha\mu}{k}, 2, 2ikr\right), \tag{6.8}$$

where $k = \sqrt{2\mu E}$ and $F(a, b, z)$ is a confluent hypergeometric function [18]. The wave function $\Psi_E(\vec{r})$ is normalized by

$$\int d^3x \Psi_E^*(\vec{r}) \Psi_E(\vec{r}) = \delta(E' - E). \tag{6.9}$$

At $\vec{r} = 0$, we get [17,23]

$$|\Psi_E(0)|^2 = \frac{1}{4\pi} \frac{\alpha\mu^2}{1 - e^{-2\pi\alpha\mu/k}}, \quad (6.10)$$

Substituting (6.10) into (6.7) and changing variables $E \rightarrow k^2/2\mu$, we obtain

$$\begin{aligned} \delta\epsilon_{1s}^{(cs)} = & -\frac{2\pi}{9} \alpha \left(2a_0^{1/2} + a_0^{3/2}\right)^2 |\Psi_{1s}(0)|^2 \\ & \times \int_0^\infty \frac{dk k}{k^2 + \alpha^2\mu^2} \frac{1}{1 - e^{-2\pi\alpha\mu/k}}. \end{aligned} \quad (6.11)$$

The integral over k is divergent. As has been shown in [11], it should be regularized by a cut-off $K = \alpha\mu$. A divergent part can be removed by a renormalization of the reduced mass of the π^-p pair. The regularized contribution of the continuous spectrum to the shift of the energy level of the ground state of pionic hydrogen reads

$$\begin{aligned} \delta\epsilon_{1s}^{(cs)} = & -\frac{2\pi}{9} \alpha \left(2a_0^{1/2} + a_0^{3/2}\right)^2 |\Psi_{1s}(0)|^2 \\ & \times \int_0^{\alpha\mu} \frac{dk k}{k^2 + \alpha^2\mu^2} \frac{1}{1 - e^{-2\pi\alpha\mu/k}}. \end{aligned} \quad (6.12)$$

Dropping the contribution of the exponential, which is insignificant in the physical region of relative momenta⁴, we get

$$\begin{aligned} \delta\epsilon_{1s}^{(cs)} = & -\frac{\pi}{9} \alpha \ell n 2 \left(2a_0^{1/2} + a_0^{3/2}\right)^2 |\Psi_{1s}(0)|^2 = \\ & -\frac{\ell n 2}{9} \alpha^4 \mu^3 \left(2a_0^{1/2} + a_0^{3/2}\right)^2. \end{aligned} \quad (6.13)$$

Thus, the total correction $\delta\epsilon_{1s} = \delta\epsilon_{1s}^{(ds)} + \delta\epsilon_{1s}^{(cs)}$ to the shift of the energy level of the ground state, caused by the $ns \rightarrow 1s$ transitions and the continuous spectrum of the π^-p pair coupled by the attractive Coulomb field in the S -wave state, is equal to

$$\begin{aligned} \delta\epsilon_{1s} = & -\frac{2}{9} \left(1 + \frac{1}{2} \ell n 2\right) \alpha \left(2a_0^{1/2} + a_0^{3/2}\right)^2 |\Psi_{1s}(0)|^2 = \\ & = -\frac{2}{9} \left(1 + \frac{1}{2} \ell n 2\right) \alpha^4 \mu^3 \left(2a_0^{1/2} + a_0^{3/2}\right)^2. \end{aligned} \quad (6.14)$$

Comparing $\delta\epsilon_{1s}$ with the DGBT formula we obtain

$$\begin{aligned} \frac{\delta\epsilon_{1s}}{\epsilon_{1s}} = & \frac{\alpha}{3} \left(1 + \frac{1}{2} \ell n 2\right) \mu \left(2a_0^{1/2} + a_0^{3/2}\right) = \\ & 0.76 \times 10^{-3} = 0.076\%, \end{aligned} \quad (6.15)$$

where we have used the experimental values of the S -wave scattering lengths $a_0^{1/2}$ and $a_0^{3/2}$ [2]:

$$\begin{aligned} a_0^{1/2} & = (+0.1788 \pm 0.0043) m_\pi^{-1}, \\ a_0^{3/2} & = (-0.0927 \pm 0.0085) m_\pi^{-1}, \\ 2a_0^{1/2} + a_0^{3/2} & = (+0.2649 \pm 0.0121) m_\pi^{-1}. \end{aligned} \quad (6.16)$$

⁴ A maximum value the exponential acquires at the upper limit. Setting $k = \alpha\mu$, one gets $e^{-2\pi\alpha\mu/k} = 1.87 \times 10^{-3}$ that is less than 0.2%.

As has been calculated in [11], the contribution to the shift of the energy level of the ground state to the second order in strong low-energy interactions $\epsilon_{1s}^{(2)}$ relative to the DGBT result is equal to

$$\begin{aligned} \delta_{1s}^{(2)} = & \frac{\epsilon_{1s}^{(2)}}{\epsilon_{1s}} = \frac{2\alpha}{\pi} \mu \frac{2 \left(a_0^{1/2}\right)^2 + \left(a_0^{3/2}\right)^2}{2a_0^{1/2} + a_0^{3/2}} = \\ & (1.11 \pm 0.06) \times 10^{-3} = (0.111 \pm 0.006)\%. \end{aligned} \quad (6.17)$$

Therefore, a total contribution to the shift of the energy level of the ground state of pionic hydrogen caused by strong low-energy interactions makes up $(0.187 \pm 0.007)\%$. Hence, it does not exceed the experimental accuracy 0.2% of the new set of experiments by the PSI Collaboration [1].

7 Conclusion

Within a quantum field theoretic, relativistic covariant and model-independent approach, we have derived the energy level displacement of the excited $n\ell$ state of pionic hydrogen in terms of the partial-wave scattering lengths of πN scattering. We have given the explicit calculation of the energy level displacements of the excited np states in terms of the P -wave scattering lengths of π^-p scattering. The shift of the energy level of the excited np state is found in analytical agreement with the result obtained by Ericson and Weise [9]. We have shown that the contributions of the energy level displacements of the np states to the transitions $np \rightarrow 1s$ are much less than the experimental accuracy of 0.2% [1]. Therefore, they can be neglected for the extraction of the experimental value of the energy level displacement of the ground state of pionic hydrogen from the $np \rightarrow 1s$ transitions.

We have given numerical values only for the energy level displacements of the excited np states. These numerical values are needed for the theoretical elaboration of experimental data on the $np \rightarrow 1s$ transitions of pionic hydrogen, used by the PSI Collaboration for the measurements of the energy level displacement of the ground state of pionic hydrogen [1]. Experimentally, there are measured only scattering lengths of πN scattering in the P ($\ell = 1$), D ($\ell = 2$) and F ($\ell = 3$) wave states [21]. The D - and F -wave scattering lengths are by factors of 10 and 100, respectively, smaller compared with the P -wave scattering lengths [21]. Hence, the energy level displacements of the excited $n\ell$ states for $\ell \geq 2$ are negligibly smaller compared with the energy level displacements of the excited np states. It is obvious that for the contemporary level of accuracy of experimental technique the energy level displacements of the excited $n\ell$ states with $\ell \geq 1$ cannot be practically measured, and the contributions of them should be neglected for the extraction of the experimental value of the energy level displacement of the ground state of pionic hydrogen from the $n\ell \rightarrow 1s$ transitions.

We have calculated the contribution of the $ns \rightarrow 1s$ transitions and the continuous spectrum of the π^-p pair, coupled by the attractive Coulomb field in the S -wave

state, to the shift of the energy level of the ground state of pionic hydrogen induced by strong low-energy interactions. The numerical value of this contribution relative to the DGBT expression makes up 0.076%. Taking into account the result obtained in [11], the total shift of the energy level of the ground state of pionic hydrogen to the second order of perturbation theory in strong low-energy interactions makes up 0.187%. This does not exceed the experimental accuracy 0.2% of the new set of experiments on the energy level displacement of the ground state of pionic hydrogen by the PSI Collaboration [1].

The obtained results confirm that the contributions of QCD isospin-breaking and electromagnetic interactions, calculated by Gasser *et al.* [12] (see also [8]), are the most important for the precise extraction of the S -wave scattering lengths of πN scattering from the experimental data on the energy level displacement of the ground state of pionic hydrogen.

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Appendix A. Calculation of momentum integral in (5.1)

In this appendix we give an explicit calculation of the momentum integral in (5.1). The wave function $\Phi_{np}(k)$ is defined by (2.10). Substituting (2.10) in the momentum integral in (5.1), we get

$$\begin{aligned} & \int \frac{d^3k}{(2\pi)^3} \sqrt{\frac{m_{\pi^-} m_p}{E_{\pi^-}(k) E_p(k)}} k \Phi_{np}(k) = \\ & \frac{1}{\pi^{3/2}} \int_0^\infty dr r^2 R_{n1}(r) \int_0^\infty dk \sqrt{\frac{m_{\pi^-} m_p}{E_{\pi^-}(k) E_p(k)}} k^3 j_1(kr). \end{aligned} \quad (\text{A.1})$$

For the spherical Bessel function $j_1(kr)$ we use the expression [24]

$$j_1(kr) = -\frac{d}{dr} \frac{1}{r} \left(\frac{\sin kr}{k^2} \right). \quad (\text{A.2})$$

The integration over k we carry out in the limit $m_p \rightarrow \infty$. This yields

$$\begin{aligned} & \int_0^\infty dk \sqrt{\frac{m_{\pi^-} m_p}{E_{\pi^-}(k) E_p(k)}} k^3 j_1(kr) = \\ & \int_0^\infty dk \sqrt{\frac{m_{\pi^-}}{E_{\pi^-}(k)}} k^3 j_1(kr) = \\ & \sqrt{m_{\pi^-}} \frac{d}{dr} \frac{1}{r} \frac{d}{dr} \int_0^\infty \frac{dk \cos(kr)}{(m_{\pi^-}^2 + k^2)^{1/4}} = \\ & m_{\pi^-} \frac{2^{1/4} \sqrt{\pi}}{\Gamma(1/4)} \frac{d}{dr} \frac{1}{r} \frac{d}{dr} (m_{\pi^-} r)^{-1/4} K_{1/4}(m_{\pi^-} r), \end{aligned} \quad (\text{A.3})$$

where we have used the formula [25]

$$K_\nu(xz) = \Gamma\left(\nu + \frac{1}{2}\right) \frac{(2z)^\nu}{\sqrt{\pi} x^\nu} \int_0^\infty \frac{\cos(xt) dt}{(t^2 + z^2)^{\nu+1/2}}. \quad (\text{A.4})$$

Due to the appearance of the McDonald function $K_{1/4}(m_{\pi^-} r)$ the integrand of the integral over r is localized around $r \sim 1/m_{\pi^-}$. This allows to take the wave function $R_{n1}(r)$ equal to

$$R_{n1}(r) = r \frac{2}{3} \frac{\sqrt{n^2 - 1}}{n^{5/2} a_B^{5/2}}. \quad (\text{A.5})$$

Substituting (A.3) and (A.5) in (A.1) we get

$$\begin{aligned} & \int \frac{d^3k}{(2\pi)^3} \sqrt{\frac{m_{\pi^-} m_p}{E_{\pi^-}(k) E_p(k)}} k \Phi_{np}(k) = \frac{1}{3\pi} \frac{\sqrt{n^2 - 1}}{n^{5/2} a_B^{5/2}} \frac{2^{5/4}}{\Gamma(1/4)} \\ & \times \int_0^\infty dx x^3 \frac{d}{dx} \left(\frac{1}{x} \left(\frac{d}{dx} x^{-1/4} K_{1/4}(x) \right) \right), \end{aligned} \quad (\text{A.6})$$

where $x = m_{\pi^-} r$. Using the relation [26]

$$\frac{1}{x} \left(\frac{d}{dx} x^{-1/4} K_{1/4}(x) \right) = -x^{-5/4} K_{5/4}(x), \quad (\text{A.7})$$

we transform the integral over x to the form

$$\begin{aligned} & \int_0^\infty dx x^3 \frac{d}{dx} \left(\frac{1}{x} \left(\frac{d}{dx} x^{-1/4} K_{1/4}(x) \right) \right) = \\ & - \int_0^\infty dx x^3 \frac{d}{dx} \left(x^{-5/4} K_{5/4}(x) \right) = \\ & 3 \int_0^\infty dx x^{3/4} K_{5/4}(x) = 3 \cdot 2^{-1/4} \Gamma\left(\frac{3}{2}\right) \Gamma\left(\frac{1}{4}\right), \end{aligned} \quad (\text{A.8})$$

where we have used the formula [27]

$$\int_0^\infty dx x^\mu K_\nu(x) = 2^{\mu-1} \Gamma\left(\frac{\mu + \nu + 1}{2}\right) \Gamma\left(\frac{\mu - \nu + 1}{2}\right). \quad (\text{A.9})$$

Substituting (A.8) in (A.6), we get

$$\int \frac{d^3k}{(2\pi)^3} \sqrt{\frac{m_{\pi^-} m_p}{E_{\pi^-}(k) E_p(k)}} k \Phi_{np}(k) = \sqrt{\frac{n^2 - 1}{\pi n^5 a_B^5}}. \quad (\text{A.10})$$

Using this result, we calculate the energy level displacement of the excited np state of pionic hydrogen (see sect. 5).

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